Topic 6 -

Second order linear ODEs

Theory

Topic 6 – Theory of second order linear ODEs

So far we've been solving first order equations.

Now we switch to second order. We will look at these:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

(2nd order linear)

To do this we need some preliminaries.

Def: Let I be an interval. Let f_1 and f_2 be defined on I. We say that f_1 and f_2 are linearly dependent if either

① $f_1(x) = c f_2(x)$ for all x in I

or

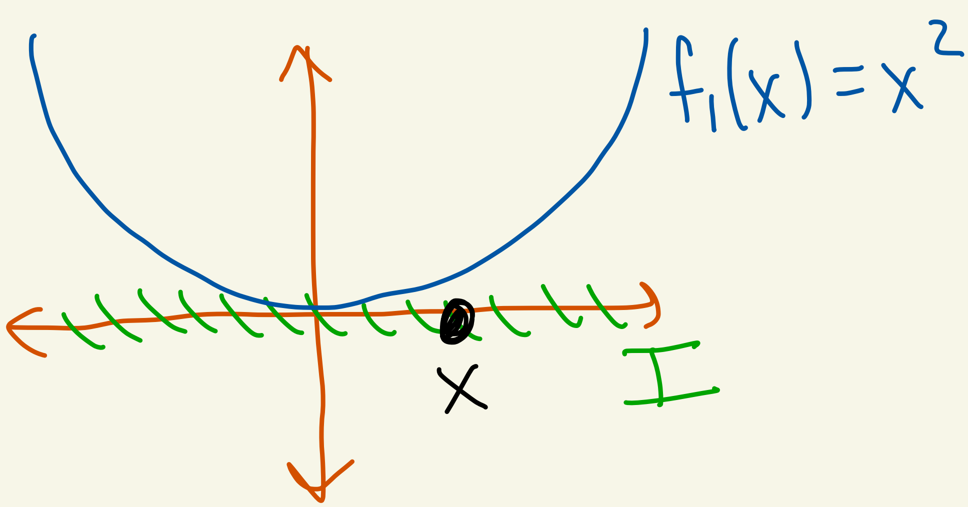
② $f_2(x) = c f_1(x)$ for all x in I

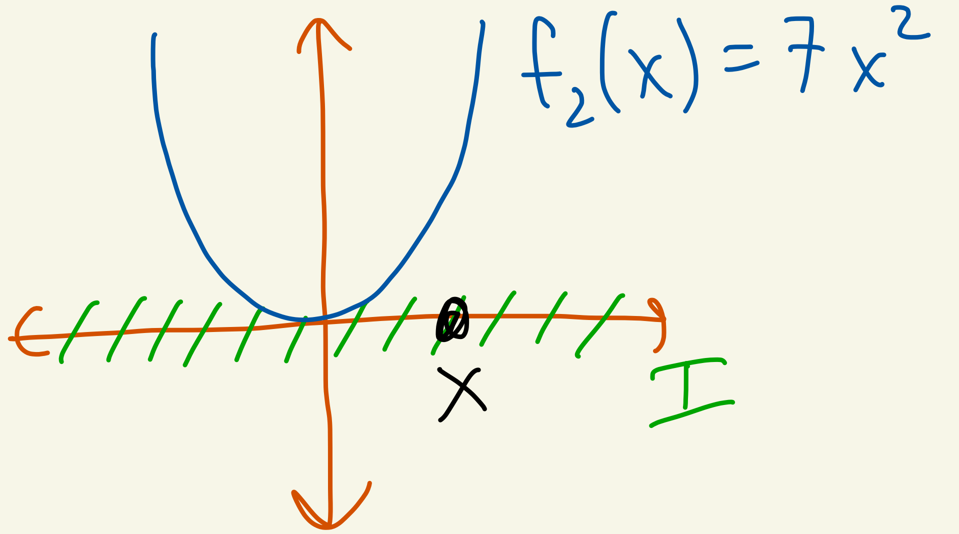
where c is a constant.

If no such c exists then f_1, f_2 are called linearly independent.

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = x^2$ and $f_2(x) = 7x^2$.





f_1 and f_2 are linearly dependent because for example

$f_1(x) = \frac{1}{7} f_2(x)$

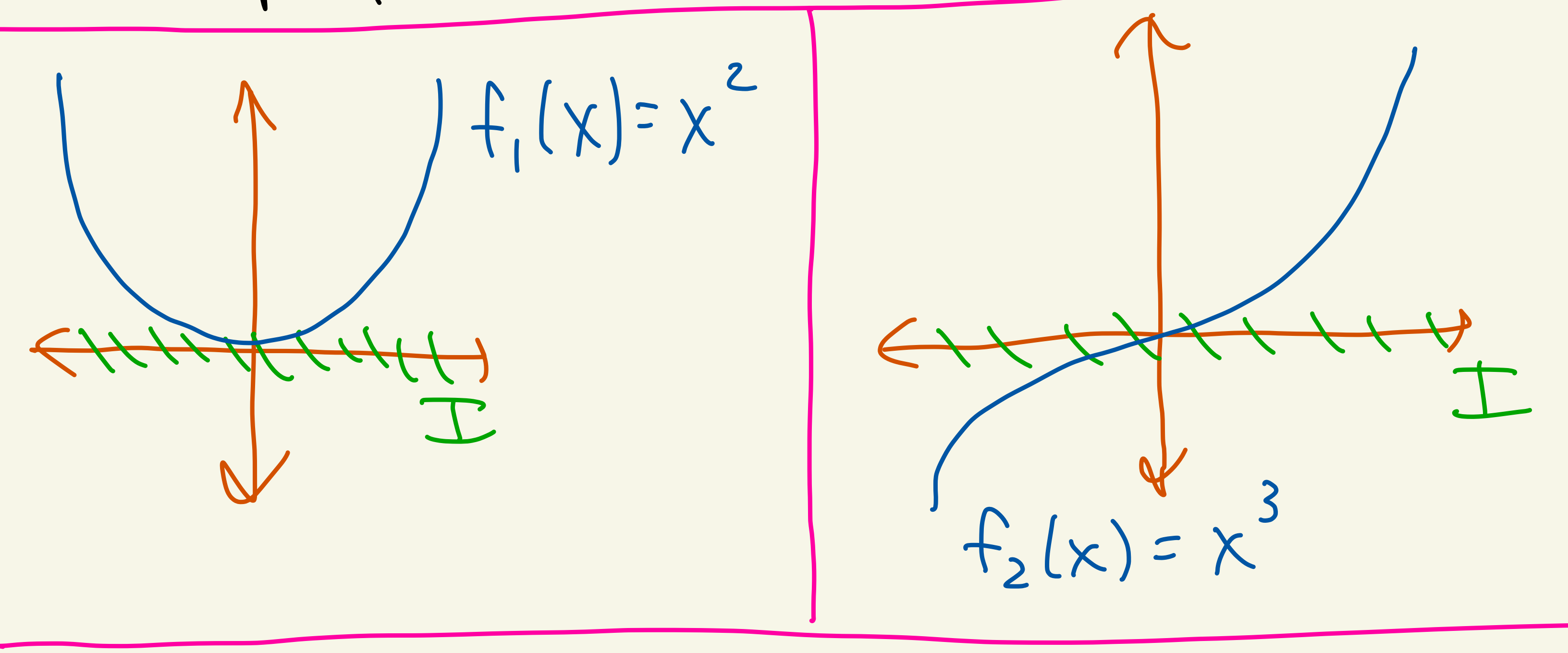
for all x in I.

$\boxed{\text{Or you could say } f_2(x) = 7 f_1(x) \text{ for all } x \text{ in } I}$

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = x^2$ and $f_2(x) = x^3$.

These functions are linearly independent. Why?



Suppose $f_1(x) = c f_2(x)$ for all x in I.

Then $x^2 = c x^3$ for all x.

Plug in $x = 1$ to get $1 = c$.

Plug in $x = 2$ to get $\frac{1}{2} = c$

This is nonsense!

Similarly you cant have $f_2(x) = c f_1(x)$.

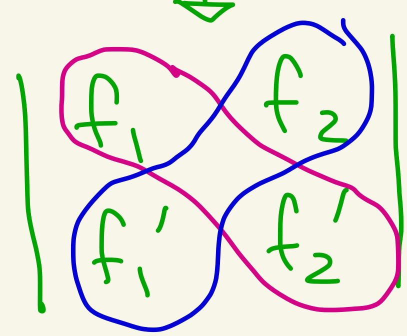
They must be linearly independent!

We will learn another way to check this based on the Wronskian.

Josef Wronski (1778-1853)

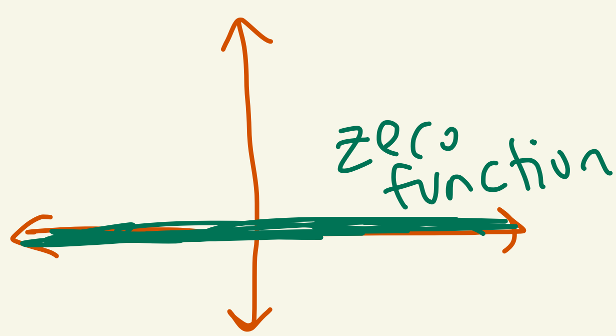
<u>Theorem:</u> Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian

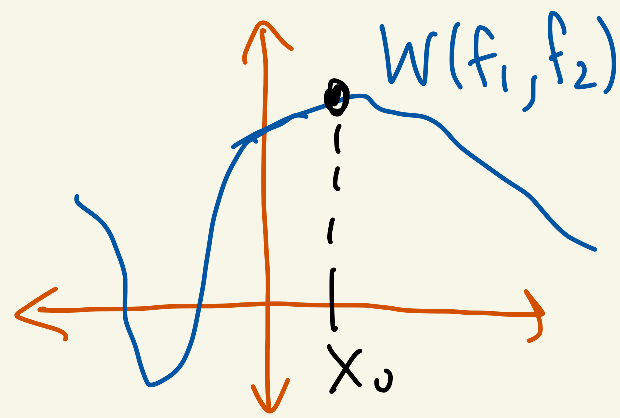
$$\underbrace{W(f_1,f_2)}_{\text{notation}} = \underbrace{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}_{\text{notation for determinant}} = \underbrace{f_1 f_2' - f_2 f_1'}_{\text{picture}}$$



is not the zero function, then f_1 and f_2 are linearly independent.

That is, if there exists an x_0 in I with $W(f_1,f_2)(x_0) \neq 0$ then f_1, f_2 are linearly independent





Ex: Let $I=(-\infty,\infty)$ and $f_1(x)=e^{2x}$, $f_2(x)=e^{5x}$.

Let's show these functions are linearly independent.

$$W(f_1,f_2)=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$=\begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

$$=(e^{2x})(5e^{5x})-(2e^{2x})(e^{5x})$$

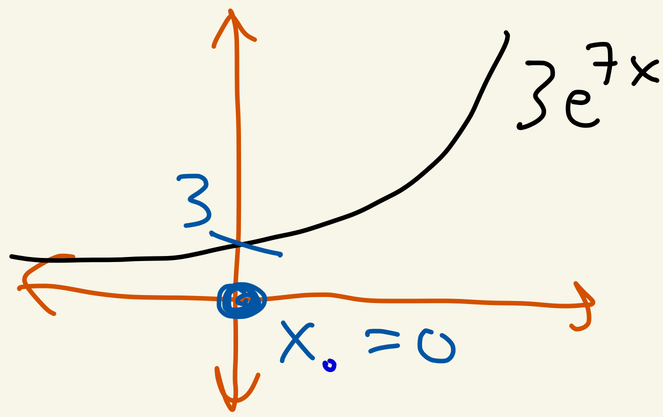
$$=5e^{7x}-2e^{7x}$$

$$=3e^{7x}$$

is this the zero function?

Plug in $x_0 = 0$

to get $3e^{7(0)} = 3 \neq 0$



Since the Wronskian is not the zero function, f_1 and f_2 are linearly independent.

For the remainder of topic 6 we will be learning the theory of solving 2nd order linear ODE

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)}$$

on some interval I where $a_2(x), a_1(x), a_0(x), b(x)$ are continuous on I and $a_2(x) \neq 0$ on I. We will assume these conditions for the rest of topic 6.

Ex: $x^2 y'' - \underline{4x} y' + 6y = \frac{1}{x}$

$a_2(x) = x^2$ $\quad a_1(x) = -4x$ $\quad a_0(x) = 6$ $\quad b(x) = \frac{1}{x}$

$I = (0, \infty)$

$\boxed{\text{Fact 1:}}$ If $f_1(x)$ and $f_2(x)$ are linearly independent solutions to the homogeneous equation

linear

homogenous means right side is 0

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on I, then every solution to $(*)$ on I is of the form

$$\boxed{y_h = c_1 f_1(x) + c_2 f_2(x)}$$

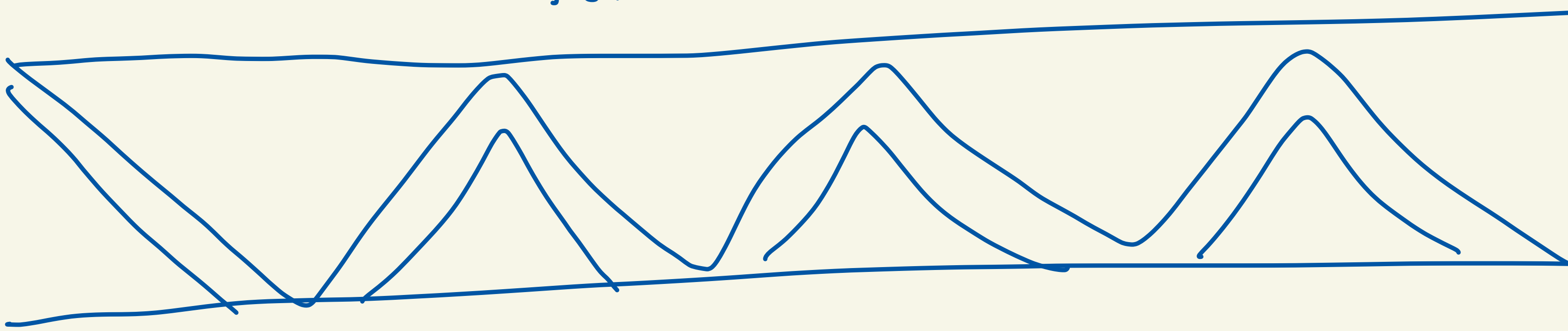
where c_1, c_2 are constants.

$\boxed{\text{Fact 2:}}$ Suppose we can find a particular solution y_p to

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)} \quad (**)$$

on I, then every solution to $(**)$ on I is of the form

$$y = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{\substack{y_h \\ \text{homogeneous} \\ \text{solution}}} + y_p$$



Ex: Let's solve

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$

Step 1: Solve the homogeneous equation:

$$y'' - 7y' + 10y = 0$$

Consider $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.
Above we showed that
f_1 and f_2 are linearly
independent. Let's show
they both solve $\boxed{y'' - 7y' + 10y = 0}$
Let's plug them in.
We have:

$f_1 = e^{2x}$, $f_1' = 2e^{2x}$, $f_1'' = 4e^{2x}$

$f_2 = e^{5x}$, $f_2' = 5e^{5x}$, $f_2'' = 25e^{5x}$

Plug them in to get:

$$f_1'' - 7f_1' + 10f_1$$
$$= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x})$$
$$= (4 - 14 + 10)e^{2x}$$

$$= 0$$

And,

$$f_2'' - 7f_2' + 10f_2$$

$$= 25e^{25} - 7(5e^{5x}) + 10(e^{5x})$$

$$= (25 - 35 + 10)e^{5x}$$

$$= 0$$

Summary: Since f_1 and f_2 are linearly independent solutions to $y'' - 7y' + 10y = 0$ that means that all solutions to $y'' - 7y' + 10y = 0$ are of the form

$$y_h = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{c_1 f_1 + c_2 f_2}$$

where c_1, c_2 are any constants

Some example solutions to $y''-7y'+10y=0$ are:

$c_1=1, c_2=7$: $y=e^{2x}+7e^{5x}$

$c_1=0, c_2=1$: $y=e^{5x}$

$c_1=\frac{1}{2}, c_2=\pi$: $y=\frac{1}{2}e^{2x}+\pi e^{5x}$

<u>Step 2:</u> Let's now solve

$$y''-7y'+10y=24e^{x}$$

on $I=(-\infty,\infty)$.

Consider

$$y_p=6e^{x}$$

We will learn how to find this later

Let's verify that y_p solves

$$y''-7y'+10y=24e^{x}$$

We get

$$y_p = 6e^x, \quad y_p' = 6e^x, \quad y_p'' = 6e^x$$

So,

$$\begin{aligned} & y_p'' - 7y_p' + 10y_p \\ &= 6e^x - 7(6e^x) + 10(6e^x) \\ &= (6 - 42 + 60)e^x \\ &= 24e^x \end{aligned}$$

Answer: Every solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$, is of the form

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{\substack{\text{general solution } y_h \\ \text{to the homogeneous} \\ y'' - 7y' + 10y = 0}} + \underbrace{6e^x}_{\substack{\text{particular} \\ \text{solution } y_p \text{ to} \\ y'' - 7y' + 10y = 24e^x}}$$

Ex: Let's find all the solutions to

$$\boxed{x^2 y'' - 4xy' + 6y = \frac{1}{x}}$$

on $I = (0, \infty)$

Step 1: First solve the homogeneous equation

$$x^2 y'' - 4xy' + 6y = 0$$

Consider

$f_1(x) = x^2$

$f_2(x) = x^3$

We will see how to find these later

First we check that f_1, f_2 are linearly independent.

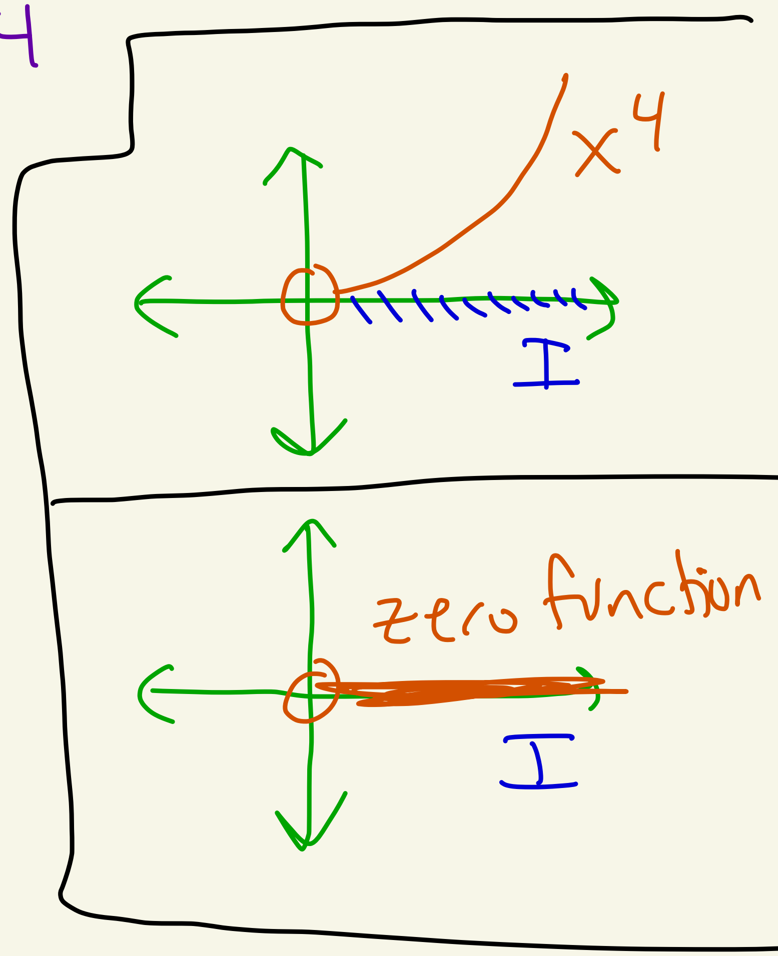
We have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= (x^2)(3x^2) - (2x)(x^3)$$

$$= x^4$$



This is not the zero function on $I = (0, \infty)$.

So, f_1, f_2 are linearly independent on $I = (0, \infty)$

Now we check that f_1, f_2 solve $x^2y'' - 4xy' + 6y = 0$.

We have

$f_1 = x^2$, $f_1' = 2x$, $f_1'' = 2$

$f_2 = x^3$, $f_2' = 3x^2$, $f_2'' = 6x$

Plugging in we get:

$$x^2 f_1'' - 4x f_1' + 6 f_1$$
$$= x^2(2) - 4x(2x) + 6(x^2)$$
$$= 0$$

And,

$$x^2 f_2'' - 4x f_2' + 6 f_2$$
$$= x^2(6x) - 4x(3x^2) + 6(x^3)$$

$= 0$

Summary: Since f_1 and f_2 are linearly independent solutions to

$$x^2y'' - 4xy' + 6y = 0$$

on I, we know every solution on I is of the form

$$y_h = \underbrace{c_1 x^2 + c_2 x^3}_{c_1 f_1 + c_2 f_2}$$

where c_1, c_2 are any constants

Step 2: Now we need a particular solution y_p to

$$x^2y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$.

Let's try

$$y_p = \frac{1}{12x} = \frac{1}{12}x^{-1}$$

Will find in topic 10

We plug it in.

$$y_p = \frac{1}{12}x^{-1}$$

$$y_p' = -\frac{1}{12}x^{-2}$$

$$y_p'' = \frac{2}{12}x^{-3} = \frac{1}{6}x^{-3}$$

We have:

$$x^2 y_p'' - 4x y_p' + 6y_p$$

$$= x^2\left(\frac{1}{6}x^{-3}\right) - 4x\left(-\frac{1}{12}x^{-2}\right) + 6\left(\frac{1}{12}x^{-1}\right)$$

$$= \frac{1}{6}x^{-1} + \frac{1}{3}x^{-1} + \frac{1}{2}x^{-1}$$

$$= x^{-1} = \frac{1}{x}$$

It's a solution!

Answer: Every solution to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ is of the form

$$y = \underbrace{c_1 x^2 + c_2 x^3}_{\text{general solution } y_h \text{ to homogeneous}} + \underbrace{\frac{1}{12}x^{-1}}_{\text{particular solution } y_p \text{ to } x^2y''-4xy'+6y=\frac{1}{x}}$$

$x^2y''-4xy'+6y=0$

Ex: Above, we showed that the general solution to

$$y''-7y'+10y=24e^x$$

on $I=(-\infty,\infty)$ is

$$y=\underbrace{c_1e^{2x}+c_2e^{5x}}_{y_h}+\underbrace{6e^x}_{y_p}$$

where c_1, c_2 are any constants.
So we get an infinite # of solutions to the differential equation, some solutions are:

$$y=\underbrace{0e^{2x}+0e^{5x}}_{c_1=0,\,c_2=0}+6e^x=6e^x$$

$$y = \underbrace{e^{2x} - 12e^{5x}}_{c_1 = 1,\ c_2 = -12} + 6e^{x}$$

However, if you create an initial-value problem by specifying $y(x_0) = y_0$, $y'(x_0) = y_0'$ at some x_0, then there will only be one unique solution!

Ex: Solve

$$y'' - 7y' + 10y = 24e^{x}$$

$$y(0) = 0, \quad y'(0) = 1$$

$x_0 = 0$

The general solution to

$$y'' - 7y' + 10y = 24e^{x}$$

is

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

Let's make this also solve

$$y(0) = 0 \text{ and } y'(0) = 1$$

We have

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$

Need to solve:

$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^{0} &= 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^{0} &= 1 \end{aligned}}$$

$e^0 = 1$

$$\boxed{\begin{aligned} c_1 + c_2 + 6 &= 0 \\ 2c_1 + 5c_2 + 6 &= 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= -5 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$.

Plug this into (2) to get:

$$2(-6 - c_2) + 5c_2 = -5$$

So, $-12 - 2c_2 + 5c_2 = -5$

Thus, $3c_2 = 7$

So, $\boxed{c_2 = 7/3}$

c_1

Then, $c_1 = -6 - c_2 = -6 - 7/3 = \boxed{-25/3}$

This gives

$$\boxed{y = -\frac{25}{3}e^{2x} + \frac{7}{3}e^{5x} + 6e^{x}}$$

$c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$

This is the unique solution to the initial-value problem

$$y'' - 7y' + 10y = 24e^{x}$$
$$y(0) = 0, \quad y'(0) = 1$$

The following are proofs of some of the previus theorems for those that are interested.
We won't cover this in class
It's mostly for me :)
You would need some linear algebra and proofs background to read.

Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian $W(f_1, f_2)$ is not zero for at least one point in I, then f_1 and f_2 are linearly independent on I.

proof:
Suppose f_1 and f_2 are linearly dependent on I
Then there exist c_1, c_2, not both zero, where

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x in I.
Thus,

$$c_1 f_1'(x) + c_2 f_2'(x) = 0$$

for all x in I.
So,

$$\begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we get that $\begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix}$ is not invertible for each x in I.
Thus, $W(f_1, f_2)(x) = 0$ for all x in I.

$\blacksquare$

Theorem: [Linear, homogeneous, second order ODE]

Let I be an interval.

Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \qquad (***)$$

Suppose that

- $f_1(x)$ and $f_2(x)$ are linearly independent on I, and
- $f_1(x)$ and $f_2(x)$ are both solutions to $(***)$

Then every solution to $(***)$ is of the form

$$c_1 f_1(x) + c_2 f_2(x)$$

later we will call this y_h

for some constants c_1, c_2.

proof:

By linearity, $c_1 f_1(x) + c_2 f_2(x)$ will be a solution to $(***)$.

Since f_1 and f_2 are linearly independent on I, by the previous theorem there exists t in I where $W(f_1,f_2)(t) \neq 0$.

Let Ψ be some solution of (***).

Consider the system

$$c_1 f_1(t) + c_2 f_2(t) = \Psi(t)$$
$$c_1 f_1'(t) + c_2 f_2'(t) = \Psi'(t)$$

This system will have a unique solution for c_1, c_2 since

$$W(f_1,f_2)(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} \neq 0.$$

Let $\hat{c}_1, \hat{c}_2$ be the unique solution and define

$$Z(t) = c_1 f_1(x) + c_2 f_2(x).$$

By the linearity of (***) we know Z satisfies (***). Z also satisfies the initial conditions $Z(t) = \Psi(t)$, $Z'(t) = \Psi'(t)$ from above. Since Ψ satisfies the same initial value problem, by the uniqueness theorem we have $\Psi(x) = Z(x)$ for all x in I. $\blacksquare$

Theorem: Let I be an interval.
Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.
Consider
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$
Suppose that f_1 and f_2 are linearly independent solutions to the homogeneous eqn
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
on I.
Suppose that f_p is a particular solution to
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$
on I.
Then every solution to
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$
is of the form
$$f(x) = \overbrace{c_1 f_1(x) + c_2 f_2(x)}^{y_h} + f_p(x)$$
for some constants c_1, c_2.

proof:
Let f solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$. Then, $f - f_p$ will solve the homogeneous equation. Hence $f - f_p = c_1 f_1 + c_2 f_2$ for some c_1, c_2. So, $f = c_1 f_1 + c_2 f_2 + f_p$ $\blacksquare$